Maths club (Liverpool) Combinatorics + Probability

§1.1

- $\binom{n}{r}$ = N° of ways of arranging n objects, $(n-r)$ objects of one (identical) type and r objects of another (identical) type.

  = N° of selections of r objects from n, where order of selection is not important. (objects distinct).

- e.g. $\binom{n}{n_1, n_2, n_3}$ = N° of ways to arrange n objects, where in each group n_i the objects are identical.

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§1.2

- $\binom{100}{97} = \frac{100!}{97!3!} = \frac{100 \times 99 \times 96}{6}$

  $\binom{10}{3,3,4} = \frac{10!}{3!3!4!}$

  Now just use calculator.

- ANS (resp). $5!$, $\frac{7!}{3!4!}$, $\frac{6!}{1!2!3!}$.

- $|S| = 100$, $|A| = 10$, $|B| = 10$.

  $|A \cap B| = |\{(1,1)\}| = 1$. $|A^c| = |S| - |A| = 90$,

  $|B^c| = |S| - |B| = 90$.

  $|A \cup B| = |A| + |B| - |A \cap B| = 19$,

  $|A^c \cap B^c| = |(A \cup B)^c| = |S| - |A \cup B| = 81$.

  $|A^c \cup B^c| = |(A \cap B)^c| = |S| - |A \cap B| = 99$.

  $|A \cap B^c| = |A| - |A \cap B| = 9$

  $|A \cup B^c| = |A| + |B^c| - |A \cap B^c|$

  $= 91$.

- $S = \{1,2,3,4,5\}$.
  $= \{M_1, M_2, M_3, X_1, X_2\}$, M_i are my numbers.

  $\binom{5}{3} = 10$ is N° of ways to choose 3 numbers (we can list options explicitly).

  i) We want 2 numbers from $\{M_1, M_2, M_3\}$ and 1 number from $\{X_1, X_2\}$, or 3 numbers from $\{M_1, M_2, M_3\}$.

  ANS: $\binom{3}{2}\binom{2}{1} = 6$, and $\binom{3}{3}\binom{2}{0} = 1$. Total 7.

  ii) ANS: $\dfrac{7}{\binom{5}{3}} = 0.7$.

  iii) Probability 1. (Impossible to avoid at least one M_i).

- Common mistake is to find answers of $\dfrac{1}{(365)^3}$, since birthday of 1st child is irrelevant. (The other two just have to match this). ANS: $\dfrac{1}{(365)^2}$.

- As above, mistake is to give answer $\dfrac{1}{(24\times 60)^2}$, it is just $\dfrac{1}{(24\times 60)}$.

- ANS: Consider n people and no coincident birthdays, and we obtain probability (for at least one sharing):

$$1 - \frac{364}{365} \times \frac{363}{365} \times \frac{362}{365} \times \cdots \times \frac{(365-(n-1))}{365} \overset{\text{By choice of } n.}{>} \frac{1}{2}$$

$n \approx 23$.

- $S = \{1, 2, \ldots 10\}$.
  $= \{M_1 \ldots M_4, X_1 \ldots X_6\}$.

i) ANS: $\binom{10}{4} - \binom{6}{4}$ $\left(= \left| \begin{matrix} \text{Total} \\ \text{N° of Selections of} \\ 4 \text{ from } S \end{matrix} \right| - \left| \begin{matrix} \text{N° of Selections of} \\ \text{all from } X_1 \ldots X_6 \end{matrix} \right| \right)$

ii) ANS: $\begin{pmatrix} 3 \text{ selected} \\ \text{from } M_1 \ldots M_4 \end{pmatrix} \times \begin{pmatrix} 1 \text{ selected} \\ \text{from } X_1 \ldots X_6 \end{pmatrix} = \binom{4}{3}\binom{6}{1}$.

- YOUME. Let $\Phi = \text{YOU}$, $\Psi = \text{ME}$.

  $A =$ set of arrangements of ΦME (3 objects)
  $B =$ ———" ——— ΨYOU (4 objects).

Total arrangements $= 5! = 120$.
$|A| = 3! = 6$, $|B| = 4! = 24$.
$\Rightarrow$ $|A \cap B| = 2!$ since $A \cap B =$ arrangements of $\Phi\Psi$.
and $|A \cup B| = |A| + |B| - |A \cap B| = 28$.
$|(A \cup B)^c| = \text{Total} - |A \cup B| = 92$.

<u>§1.3</u>: • $\binom{n}{r} = \binom{n}{n-r}$, <u>i.e</u> for every r selected from n, $(n-r)$ objects are left behind. This is in one-to-one correspondence.

- $S = \{a_1, a_2 \cdots a_n\} \equiv \{a_1 \cdots a_{n-1}, X\}$. X = special.

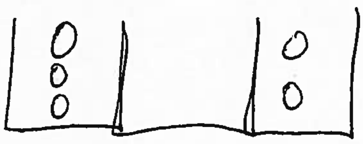
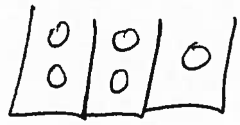
  Include X: choose $(r-1)$ from $(n-1)$.
  Exclude X: choose r from $(n-1)$.

  $$\Rightarrow \binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}.$$

  For next, let $S = \{a_1 \cdots a_{n-2}, X, Y\}$.

  Include X, Y, include X or Y (not both), exclude both X, Y.

  (or iterate the above).

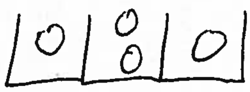
-  = 0001100,  = 0010010.

  N° of arrangements $= \dfrac{(5+2)!}{5!\,2!}$.

- In general: <u>ANS</u>: $\binom{n+r-1}{r-1}$.

- e.g. $O_{\wedge} O_{\wedge} O_{\wedge} O \ldots\ldots O_{\wedge} O$

  "$\wedge$" means insert divider. (At least one ball in each box).

  So 010010 =  4 balls
  2 dividers.

  Thus $(n-1)$ spaces for $(r-1)$ dividers.

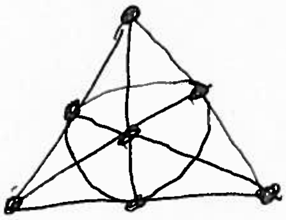
  ANS: $\binom{n-1}{r-1}$.

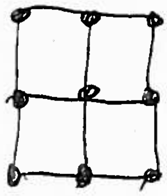
$(w+x+y+z)^5$: General term $w^a x^b y^c z^d$, $a+b+c+d=5$
(All ≥ 0).

Think of 5 balls in 4 boxes: ANS: $\binom{8}{3}$.

§1.4 Hints: $S(2,3,4)$ is NOT a Steiner system.

In general $S(2,3,n)$ is Steiner if $n \equiv 1$ or $3 \bmod 6$.

$S(2,3,7) =$  Label vertices from $\{1, \dots 7\}$.

$S(2,3,9)$, form grid , Label vertices from $\{1, \dots 9\}$.

(This gives 6 out of possible 12. The other 6 blocks are constructed using diagonal lines on $\mathbb{Z}_3 \times \mathbb{Z}_3$).

- N° of r-element sets $= \binom{n}{r}$.

  N° of r-elements per block $= \binom{k}{r}$.

Lottery: N° of r-element sets $= \binom{49}{3}$.

N° of r-elements per block $= \binom{6}{3} = 20$.

$\Rightarrow$ N° of tickets ≈ 900 (ans not an integer for B).

However any ticket bought as above (to optimize) would lead to win at least 20 times over (if 3 numbers matched), since each 6 N° draw has 20 3-element sets.

Approx answer (on web) is circa 100–200 range.

§1.5 :

- ANS 2^n possible sequences
- let S_n = admissible string of length n.

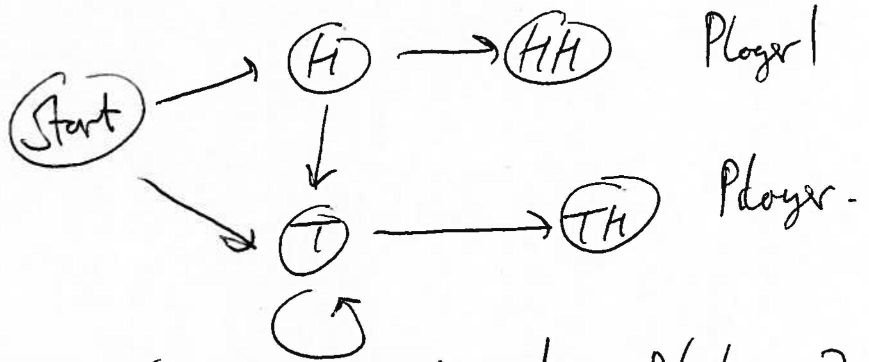
Then $S_1 = H, T$. $S_2 = HH, TH, HT$, etc.

$S_n = HS_{n-1}$ or THS_{n-2} (since TT not allowed)

$\Rightarrow |\#S_n| = |\#S_{n-1}| + |\#S_{n-2}|$,

where $\# = N^o$.

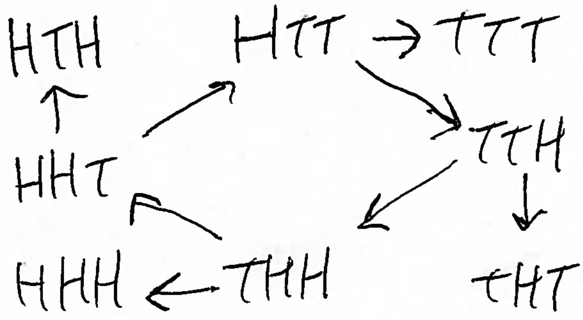
The recurrence relation above is the <u>Fibonacci sequence</u>.

- 

We see $P(\text{player 1 win}) = \frac{1}{4}$, $P(\text{player 2 win}) = \frac{3}{4}$.

for HHH versus THH, $P(\text{Player 2 win}) = \frac{7}{8}$.

for HHHH versus THHH $(\text{Player 2 win}) = \frac{15}{16}$.

- 

"Strong → Weak".

§1.6 Monty Hall :

<u>ANS</u> : <u>switch</u>! But be careful of assumptions used in variations.